Reconstructing Stiffness Function of Symmetric Two-Span Beam with Two Free Ends Using Symmetric Mode

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Keywords: symmetric mode; two-span beam with two free ends; inverse problem

Abstract: In this paper we introduced an inverse problem of symmetric beam with two free ends. When the circular frequency and the symmetric mode were given, we discussed when the density function of the symmetric beam was symmetric polynomial type function, how to construct the stiffness function for the same symmetric polynomial function.

1. Introduction

For the transverse vibration of *Euler* beam, *Euler* established the well-known *Euler* beam modal equation and gave the classical solution. In 2001, *Elishakoff I.* put forward a new solution [1,2]. The displacement function, density function and stiffness function in the modal equation of *Euler* beam are all set as polynomial functions, but they have different orders. This method is called the inverse problem of structural vibration, which can be applied to many kinds of one-dimensional elastic structures [3-7].

In Ref. [8, 9], symmetric one-dimensional elastic structures are researched. A new kind of symmetric multi-span beam, i.e. the inverse problem in vibration of symmetric two-span beam with two free ends, is researched in this paper. In daily life, teeterboard and shoulder pole can be regarded as symmetric two-span beam with two free ends.

2. Statement of the problem

The dimensionless dynamic equation for the transverse vibration of an *Euler* beam with a length of l is:

$$\frac{d^2}{d\xi^2}\left[r(\xi)\frac{d^2W(\xi)}{d\xi^2}\right]-\lambda\rho(\xi)W(\xi)=0. \tag{1}$$

In the above differential equation, the independent variable $\xi = x/l$ is a dimensionless axial coordinate, $r(\xi)=E(\xi)I(\xi)$ is stiffness function, $E(\xi)$ is Young's modulus, $I(\xi)$ is moment of inertia of a section, $W(\xi)$ is displacement mode, $\lambda=\omega^2 Al^4$ is eigenvalue for this problem, ω is circular frequency, A is cross-sectional area, $\rho(\xi)$ is density function. To the symmetric two-span beam with two free ends, there is a support at the mid-point $\xi = 1/2$ as shown in *Figure* 1.

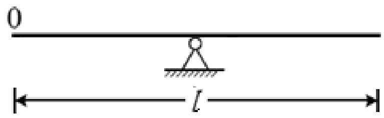

Fig. 1 Symmetric Two-Span Beam with Two Free Ends

To the symmetric mode of symmetric two-span beam with two free ends, there are the following boundary conditions [10] and constraint condition:

$$r(0)W''(0)=0, \qquad [r(0)W''(0)]'=0, \qquad r(1)W''(1)=0, \qquad [r(1)W''(1)]'=0, \tag{2}$$

$$W(\tfrac{1}{2}) = 0. \tag{3}$$

Since when Young's modulus is zero, there is no physical meaning, so the equation (2) is equivalent to

$$W''(0) = 0, \qquad W'''(0) = 0, \qquad W''(1) = 0, \qquad W'''(1) = 0. \tag{4}$$

From the symmetry of mode, there is

$$W(0) = W(1). \tag{5}$$

To meet the requirements (3), (4) and (5), displacement polynomial can be taken as six-order polynomial. It is assumed that

$$W(\xi) = W_0 + W_1\xi + W_2\xi^2 + W_3\xi^3 + W_4\xi^4 + W_5\xi^5 + W_6\xi^6. \tag{6}$$

The displacement polynomial satisfying the requirement conditions is obtained as:

$$W^{sy}(\xi) = B(11 - 32\xi + 160\xi^4 - 192\xi^5 + 64\xi^6). \tag{7}$$

Here, $B = W_6 / 64$ is arbitrary nonzero constant, superscript sy represents symmetric mode of vibration.

By simplification, the formula (7) can be reduced to the following symmetric polynomial function:

$$W^{sy}(\xi) = B[11 - 32\xi(1-\xi) - 32\xi^2(1-\xi)^2 - 64\xi^3(1-\xi)^3]. \tag{8}$$

According to Ref. [8], the density function $\rho(\xi)$ and stiffness function $r(\xi)$ of symmetric two-span beam with two free ends which are symmetric to mid-point $\xi = 1/2$ can be set to symmetric polynomial functions as:

$$\rho(\xi) = \sum_{i=0}^{m} a_i \xi^i (1-\xi)^i, \quad r(\xi) = \sum_{i=0}^{n} b_i \xi^i (1-\xi)^i. \tag{9}$$

In the above formulas, m and n are positive integers. The first item on the left side of equation (1) contains the four-order derivative, and therefore $n = m + 2$.

In this paper, when the density function $\rho(\xi)$, displacement mode $W^{sy}(\xi)$ and inherent circular frequency ω are all known, the stiffness function $r(\xi)$ can be resolved.

3. Stiffness function of symmetric two-span beam with two free ends from symmetric mode and given density function

It can be verified that:

$$[\xi^k(1-\xi)^k]'' = k(k-1)\xi^{k-2}(1-\xi)^{k-2} - 2k(2k-1)\xi^{k-1}(1-\xi)^{k-1}, \ \ (k = 2,3,\cdots). \tag{10}$$

From formula (7), it can be calculated as the result of:

$$W''^{sy}(\xi) = 1920B\xi^2(1-\xi)^2. \tag{11}$$

When the formulas $\rho(\xi)$, $r(\xi)$ and $W''^{sy}(\xi)$ substitute into Equation (1), and eliminate the nonzero constant B, we obtain:

$$1920\left[\sum_{i=0}^{m+2} b_i \xi^i (1-\xi)^i \xi^2 (1-\xi)^2\right]'' = \lambda \sum_{i=0}^{m} a_i \xi^i (1-\xi)^i \left[11 - 32\xi(1-\xi) - 32\xi^2(1-\xi)^2 - 64\xi^3(1-\xi)^3\right]. \tag{12}$$

Note $t = \xi(1-\xi)$, from Equation (10), we obtain:

$$\sum_{i=0}^{m+2} b_i[(i+2)(i+1)t^i - (2i+4)(2i+3)t^{i+1}] = \frac{\lambda}{1920}\sum_{i=0}^{m} a_i t^i (11 - 32t - 32t^2 - 64t^3). \tag{13}$$

For arbitrary $\xi \in [0,\ 1]$, i.e. $t \in [0,\ 0.25]$, Equation (13) is established. By using the comparison coefficient method of polynomials, we can obtain:

$$(2\times 1)b_0 = 11a_0\lambda/1920, \tag{14}$$

$$-(4\times 3)b_0 + (3\times 2)b_1 = (-32a_0 + 11a_1)\lambda/1920, \tag{15}$$

$$-(6\times 5)b_1 + (4\times 3)b_2 = (-32a_0 - 32a_1 + 11a_2)\lambda/1920, \tag{16}$$

$$-(2i+2)(2i+1)b_{i-1} + (i+2)(i+1)b_i = (-64a_{i-3} - 32a_{i-2} - 32a_{i-1} + 11a_i)\lambda/1920, \quad (i = 3,4,\cdots,m), \tag{17}$$

$$-(2m+4)(2m+3)b_m + (m+3)(m+2)b_{m+1} = (-64a_{m-2} - 32a_{m-1} - 32a_m)\lambda/1920, \tag{18}$$

$$-(2m+6)(2m+5)b_{m+1} + (m+4)(m+3)b_{m+2} = (-64a_{m-1} - 32a_m)\lambda/1920, \tag{19}$$

$$-(2m+8)(2m+7)b_{m+2} = (-64a_m)\lambda/1920. \tag{20}$$

The above equations can be written as matrix equation as:

$$1920\boldsymbol{D}\cdot\boldsymbol{B} = \lambda\boldsymbol{C}\cdot\boldsymbol{A}. \tag{21}$$

Here,

$$\boldsymbol{D} = \begin{bmatrix} 2\times 1 & 0 & 0 & 0 & \cdots & 0 \\ -4\times 3 & 3\times 2 & 0 & 0 & \cdots & 0 \\ 0 & -6\times 5 & 4\times 3 & 0 & \cdots & 0 \\ 0 & 0 & -8\times 7 & 5\times 4 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -(2m+2)(2m+1) & (m+2)(m+1) & 0 & 0 \\ 0 & \cdots & 0 & -(2m+4)(2m+3) & (m+3)(m+2) & 0 \\ 0 & \cdots & 0 & 0 & -(2m+6)(2m+5) & (m+4)(m+3) \\ 0 & \cdots & 0 & 0 & 0 & -(2m+8)(2m+7) \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 11 & 0 & 0 & 0 & \cdots & 0 \\ -32 & 11 & 0 & 0 & \cdots & 0 \\ -32 & -32 & 11 & 0 & \cdots & 0 \\ -64 & -32 & -32 & 11 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -64 & -32 & -32 & 11 \\ 0 & \cdots & 0 & -64 & -32 & -32 \\ 0 & \cdots & 0 & 0 & -64 & -32 \\ 0 & \cdots & 0 & 0 & 0 & -64 \end{bmatrix}.$$

$$\boldsymbol{B} = (b_0, b_1, b_2, \cdots b_{m+1}, b_{m+2})^T, \quad \boldsymbol{A} = (a_0, a_1, a_2, \cdots a_{m-1}, a_m)^T.$$

Among them, $\boldsymbol{D}$ is a matrix with $m+4$ row, $m+3$ column, $\boldsymbol{C}$ is another matrix with $m+4$ row and $m+1$ column. $\boldsymbol{B}$ is a $m+3$ dimension column vector, $\boldsymbol{A}$ is another $m+1$ dimension column vector. The generalized inverse matrix $\boldsymbol{X}$ of matrix $\boldsymbol{D}$ is introduced, formula (21) changed to:

$$\boldsymbol{B} = \lambda/1920\boldsymbol{X}\cdot\boldsymbol{C}\cdot\boldsymbol{A}. \tag{22}$$

4. Example

If $m=4$, formula (21) changed to:

$$1920\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -30 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & -56 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & -90 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & -132 & 42 & 0 \\ 0 & 0 & 0 & 0 & 0 & -182 & 56 \\ 0 & 0 & 0 & 0 & 0 & 0 & -240 \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \lambda \begin{bmatrix} 11 & 0 & 0 & 0 & 0 \\ -32 & 11 & 0 & 0 & 0 \\ -32 & -32 & 11 & 0 & 0 \\ -64 & -32 & -32 & 11 & 0 \\ 0 & -64 & -32 & -32 & 11 \\ 0 & 0 & -64 & -32 & -32 \\ 0 & 0 & 0 & -64 & -32 \\ 0 & 0 & 0 & 0 & -64 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix},$$

For formula (22), using *Matlab* programming, we can obtain:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \frac{\lambda}{1920} \begin{bmatrix} 3.4868 & -0.2121 & -0.0312 & -0.0053 & -0.0010 \\ 1.5285 & 1.3973 & -0.0641 & -0.0109 & -0.0020 \\ 1.1434 & 0.8255 & 0.7561 & -0.0273 & -0.0050 \\ 0.0001 & 0.7111 & 0.5172 & 0.4737 & -0.0141 \\ 0.0000 & 0.0000 & 0.4848 & 0.3543 & 0.3245 \\ 0.0000 & 0.0000 & 0.0000 & 0.3516 & 0.2579 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.2667 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}.$$

Following the method in Ref. [9], if a_i $(i=0,1,2,3,4)$ are all positive, then $\rho(\xi)=\sum_{i=0}^{4} a_i \xi^i (1-\xi)^i$ is a positive function. It is easy to prove, when satisfying a_i $(i=0,1,2,3,4)$ are all positive and $a_{i-1} > a_i$ $(i=1,2,3,4)$, then $r(\xi)=\sum_{i=0}^{6} b_i \xi^i (1-\xi)^i$ is also a positive function, and the proof procedure will not given in details.

5. Conclusion

In this paper, the method of inverse problem in vibration of *Euler* beam is successfully generalized to a new multi-span beam structure, i.e. symmetric two-span beam with two free ends. When the symmetric mode as polynomial function and the circular frequency are known, this paper discussed how to construct the stiffness function of the beam as symmetric polynomial function when the density function of the beam is given as the symmetric polynomial function. An example is given to discuss the positive value of the density function and the stiffness function.

Acknowledgements

The Corresponding author of this article is Min He. This work is supported by Anhui Provincial Natural Science Foundation under Grant No. 1808085MA05, 1808085MA20 and Grant No. 1708085MA10.

References

[1] Elishakoff I, Candan S, 2001. Apparently first closed-form solution for vibrating, Inhomogeneous beams, International Journal of Solids and Structures, Vol. 38, No. 19.

[2] Elishakoff I, Candan S, 2001. Apparently first closed-form solution for frequencies of deter ministically and /or stochastically inhomogeneous simply supported beams, Journal of Applied Mechanics, Vol. 68, No. 3.

[3] Elishakoff, I, 2005. Eigenvalues of inhomogeneous structures: unusual closed-form solutions,

CRC Press, Boca Raton.

[4] WANG Qishen, 2003. Constructing the axial stiffness of arbitrary supported rod from fundamental mode shape, Acta Mechanica Sinica, Vol. 35, No. 3. (in Chinese)

[5] WU Lei, WANG Qishen, Elishakoff I, 2005. Semi-inverse method for axially functionally graded beams with an anti-symmetric vibration mode, Journal of Sound and Vibration. Vol. 284, No. 3.

[6] WU Lei, ZHANG Lihua, 2011. WANG Qishen, Elishakoff I, Reconstructing cantilever beams via vibration mode with a given node location, Acta Mechanica, Vol. 217, No. 1.

[7] HE Min, ZHANG Lihua, WANG Qishen, 2014. Reconstructing cross-sectional physical parameters for two-span beams with overhang using fundamental mode, Acta Mechanica, Vol. 225, No. 2.

[8] WANG Qishen, LIU Minghui, ZHANG Lihua, HE Min, 2014. Reconstruction flexural stiffness of symmetric simple supported beams with single mode, Chinese Journal of Computational Physics. Vol. 31, No. 2. (In Chinese)

[9] HE Min, ZHANG Lihua, HUANG Zhong, WANG Qishen, 2017. Reconstructing the axial stiffness of a symmetric rod with two elastic supports using single mode, Acta Mechanica. Vol. 228, No. 4.

[10] WANG Dajun, WANG Qishen, HE Beichang, 2014. Qualitative Theory in Structural Mechanics, Peking University Press, Beijing. (in Chinese)